

ON THE TRANSVERSE KHOVANOV-ROZANSKY HOMOLOGIES: GRADED MODULE STRUCTURE AND STABILIZATION

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ABSTRACT. In [9], the author proved that the Khovanov-Rozansky homology \mathcal{H}_N with potential ax^{N+1} is an invariant for transverse links in the standard contact 3-sphere. In the current paper, we study the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -module structure of \mathcal{H}_N , which leads to better understanding of the effect of stabilization on \mathcal{H}_N . As an application, we compute \mathcal{H}_N for all transverse unknots.

1. INTRODUCTION

1.1. The transverse Khovanov-Rozansky homology \mathcal{H}_N . A contact structure ξ on an oriented 3-manifold M is an oriented tangent plane distribution such that there is a 1-form α on M satisfying $\xi = \ker \alpha$, $d\alpha|_{\xi} > 0$ and $\alpha \wedge d\alpha > 0$. Such a 1-form is called a contact form for ξ . The standard contact structure ξ_{st} on S^3 is given by the contact form $\alpha_{st} = dz - ydx + xdy = dz + r^2 d\theta$.

We say that an oriented smooth link L in S^3 is transverse if $\alpha_{st}|_L > 0$. Two transverse links are said to be transverse isotopic if there is an isotopy from one to the other through transverse links.

Theorem 1.1. [1, 6, 7]

- (1) Every transverse link is transverse isotopic to a counterclockwise transverse closed braid around the z -axis.
- (2) Any smooth counterclockwise closed braid around the z -axis can be smoothly isotoped into a counterclockwise transverse closed braid around the z -axis without changing the braid word.
- (3) Two counterclockwise transverse closed braids around the z -axis are transverse isotopic if and only if the braid word of one of them can be changed into that of the other by a finite sequence of transverse Markov moves. Here, by “transverse Markov moves”, we mean the following braid moves:

- Braid group relations generated by
 - $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = \emptyset$,
 - $\sigma_i \sigma_j = \sigma_j \sigma_i$, when $|i - j| > 1$,
 - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.
- Conjugation: $\mu \rightsquigarrow \eta^{-1} \mu \eta$, where $\mu, \eta \in \mathbf{B}_m$.¹
- Positive stabilization and destabilization: $\mu (\in \mathbf{B}_m) \rightsquigarrow \mu \sigma_m (\in \mathbf{B}_{m+1})$.

In other words, all Markov moves are transverse Markov moves except the negative stabilization and destabilization $\mu (\in \mathbf{B}_m) \rightsquigarrow \mu \sigma_m^{-1} (\in \mathbf{B}_{m+1})$.

Part (1) of Theorem 1.1 was established by Bennequin in [1], part (2) is a simple observation and part (3) was proved by Orevkov, Shevchishin in [6] and independently by Wrinkle in [7]. Theorem 1.1 means that there is a one-to-one correspondence

$$\{\text{Transverse isotopy classes of transverse links}\} \longleftrightarrow \{\text{Closed braids modulo transverse Markov moves}\}.$$

So, constructing invariants for transverse links is equivalent to constructing invariants for equivalence classes of closed braids modulo transverse Markov moves. For example, for a closed braid B with writhe w of m strands, its self linking number $sl(B) = w - m$ is invariant under transverse Markov moves. So the self linking number is a transverse link invariant. See [1] for the original definition of the self linking number.

For more about transverse links, see, for example, [3].

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¹In this paper, “ \mathbf{B}_m ” means the braid group on m strands.

Using the above correspondence, the author introduced in [9] a new homological invariant \mathcal{H}_N for transverse links. \mathcal{H}_N is a variant of the Khovanov-Rozansky homology defined in [4, 5]. We call \mathcal{H}_N the N th transverse Khovanov-Rozansky homology. The following is the main result of [9].

Theorem 1.2. [9, Theorem 1.2] *Suppose $N \geq 1$. Let B be a closed braid and $\mathcal{C}_N(B)$ the chain complex defined in Definition 2.13. Then the homotopy type of $\mathcal{C}_N(B)$ does not change under transverse Markov moves. Moreover, the homotopy equivalences induced by transverse Markov moves preserve the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -grading of $\mathcal{C}_N(B)$, where the \mathbb{Z}_2 -grading is the \mathbb{Z}_2 -grading of the underlying matrix factorization and the three \mathbb{Z} -gradings are the homological, a - and x -gradings of $\mathcal{C}_N(B)$.*

Consequently, for the homology $\mathcal{H}_N(B) = H(H(\mathcal{C}_N(B), d_{mf}), d_\chi)$ of $\mathcal{C}_N(B)$ defined in Definition 2.15, every transverse Markov move on B induces an isomorphism of $\mathcal{H}_N(B)$ preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -module structure of $\mathcal{H}_N(B)$.

1.2. Module structure of $\mathcal{H}_N(B)$. The first part of the current paper is a more careful study of the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -module structure of $\mathcal{H}_N(B)$, which refines [9, Theorem 1.11] and leads to Theorem 1.4 below.

Before stating Theorem 1.4, we introduce the following notations.

Definition 1.3. Let B be a closed braid. For $(\varepsilon, i, j, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$, denote by $\mathcal{H}_N^{\varepsilon, i, j, k}(B)$ the subspace of $\mathcal{H}_N(B)$ of homogeneous elements of \mathbb{Z}_2 -degree ε , homological degree i , a -degree j and x -degree k . Replacing one of these indices by a “ \star ” means direct summing over all possible values of this index. For example:

$$\begin{aligned}\mathcal{H}_N^{\varepsilon, i, \star, k}(B) &= \bigoplus_{j \in \mathbb{Z}} \mathcal{H}_N^{\varepsilon, i, j, k}(B), \\ \mathcal{H}_N^{\varepsilon, i, \star, \star}(B) &= \bigoplus_{(j, k) \in \mathbb{Z}^{\oplus 2}} \mathcal{H}_N^{\varepsilon, i, j, k}(B).\end{aligned}$$

Similarly, for the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology $H_N(B)$ defined in [4]², we denote by $H_N^{\varepsilon, i, k}(B)$ the subspace of $H_N(B)$ of homogeneous elements of \mathbb{Z}_2 -degree ε , homological degree i and x -degree k . Again, Replacing one of these indices by a “ \star ” means direct summing over all possible values of this index.

Theorem 1.4. *Let B be a closed braid, and $(\varepsilon, i, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$. As a \mathbb{Z} -graded $\mathbb{Q}[a]$ -module,*

$$\mathcal{H}_N^{\varepsilon, i, \star, k}(B) \cong (\mathbb{Q}[a]\{sl(B)\}_a)^{\oplus l} \oplus (\mathbb{Q}[a]\{sl(B) + 2\}_a)^{\oplus (\dim_{\mathbb{Q}} H_N^{\varepsilon, i, k}(B) - l)} \oplus \left(\bigoplus_{q=1}^n \mathbb{Q}[a]/(a)\{s_q\} \right),$$

where

- $\{s\}_a$ means shifting the a -grading by s ,
- l and n are finite non-negative integers determined by B and the triple (ε, i, k) ,
- $\{s_1, \dots, s_n\} \subset \mathbb{Z}$ is a sequence determined up to permutation by B and the triple (ε, i, k) ,
- $sl(B) \leq s_q \leq c_+ - c_- - 1$ and $(N-1)s_q \leq k - 2N + 2c_-$ for $1 \leq q \leq n$, where c_{\pm} is the number of \pm crossings in B .

Remark 1.5. Note that $sl(B)$ and the number of components of B have the same parity. So, from [4], we know that $H_N^{sl(B)-1, i, k}(B) \cong 0$ and, by Theorem 1.4, $\mathcal{H}_N^{sl(B)-1, i, \star, k}(B)$ is a torsion $\mathbb{Q}[a]$ -module.

1.3. Stabilization. Applying a negative stabilization to a transverse closed braid B , we get a new transverse closed braid B_- . In contact geometry, this procedure is called a stabilization of the transverse link. In [9, Theorem 1.5], the author established that the chain complex $\mathcal{C}_N(B_-)$ is isomorphic to $\text{cone}(\pi_0)\{-2, 0\}$, where

- $\pi_0 : \mathcal{C}_N(B) \rightarrow \mathcal{C}_N(B)/a\mathcal{C}_N(B)$ is the standard quotient map,
- $\text{cone}(\pi_0)$ is the mapping cone of π_0 ,
- $\{j, k\}$ means shifting the a -grading by j and the x -grading by k .

²See Subsection 2.4 for our normalization of $H_N(B)$.

Therefore, there is a long exact sequence

$$\cdots \rightarrow \mathcal{H}_N^{\varepsilon, i-1, \star, \star}(B)\{-2, 0\} \xrightarrow{\pi_0} \mathcal{H}_N^{\varepsilon, i-1, \star, \star}(B)\{-2, 0\} \rightarrow \mathcal{H}_N^{\varepsilon, i, \star, \star}(B_-) \rightarrow \mathcal{H}_N^{\varepsilon, i, \star, \star}(B)\{-2, 0\} \xrightarrow{\pi_0} \mathcal{H}_N^{\varepsilon, i, \star, \star}(B)\{-2, 0\} \rightarrow \cdots$$

preserving the a - and x -gradings, where $\mathcal{H}_N(B) := H(H(\mathcal{C}_N(B)/a\mathcal{C}_N(B), d_{mf}), d_\chi)$.

Generally, it is not very easy to compute $\mathcal{H}_N(B)$ even if $\mathcal{H}_N(B)$ is known. So the above long exact sequence is not very useful when computing the homology of a stabilization of a transverse link. Using Theorem 1.4, we will take a closer look at the chain complex $\mathcal{C}_N(B_-) \cong \text{cone}(\pi_0)\{-2, 0\}$ and deduce Theorem 1.6 below.

Theorem 1.6. *Let B be a closed braid and B_- a stabilization of B . Set $s = \text{sl}(B)$. Then for any $(i, k) \in \mathbb{Z}^{\oplus 2}$, there are a long exact sequence of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules*

$$(1.1) \quad \cdots \rightarrow \mathcal{H}_N^{s-1, i, \star, k}(B_-) \rightarrow \mathcal{H}_N^{s, i-1, \star, k+N+1}(B)\{-1\}_a \rightarrow H_N^{s, i-1, k+N+1}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s-1\}_a \rightarrow \mathcal{H}_N^{s-1, i+1, \star, k}(B_-) \rightarrow \cdots$$

and a short exact sequence of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules

$$(1.2) \quad 0 \rightarrow H_N^{s, i, k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s\}_a \rightarrow \mathcal{H}_N^{s, i, \star, k}(B_-) \rightarrow \mathcal{H}_N^{s-1, i-1, \star, k+N+1}(B)\{-1\}_a \rightarrow 0,$$

where $H_N(B)$ is the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology of B defined in [4].

In [2], Eliashberg and Fraser showed that two transverse unknots are transverse isotopic if and only if their self linking numbers are equal. Bennequin's inequality [1] implies that the highest self linking number of a transverse unknot is -1 , which is attained by the 1-strand transverse closed braid. Denote by U_0 the transverse unknot with self linking -1 and by U_m the transverse unknot obtained from U_0 by m stabilizations. Then every transverse unknot is transverse isotopic to U_m for some $m \geq 0$.

As an application of Theorem 1.6, we compute \mathcal{H}_N for all the transverse unknots. Before stating the result, let us recall that the \mathbb{Z} -grading of $\mathbb{Q}[a]$ is given by $\deg_a a = 2$. We make $\mathbb{Q}[a]$ a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -module by making the \mathbb{Z}_2 -, homological and x -gradings all 0 on $\mathbb{Q}[a]$.

Corollary 1.7. *Let \mathcal{F} and \mathcal{T} be the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -modules*

$$\begin{aligned} \mathcal{F} &:= \bigoplus_{l=0}^{N-1} \mathbb{Q}[a]\langle 1 \rangle \{-1, -N+1+2l\}, \\ \mathcal{T} &:= \bigoplus_{l=0}^{\infty} \mathbb{Q}[a]/(a)\langle 1 \rangle \{-1, N+1+2l\}, \end{aligned}$$

where " $\langle \varepsilon \rangle$ " means shifting the \mathbb{Z}_2 -grading by ε and " $\{j, k\}$ " means shifting the a -grading by j and the x -gradings by k . Then,

$$\begin{aligned} \mathcal{H}_N(U_0) &\cong \mathcal{F} \oplus \mathcal{T}, \\ \mathcal{H}_N(U_1) &\cong \mathcal{F} \oplus \mathcal{T}\langle 1 \rangle \{-1, -N-1\} \| 1 \|, \end{aligned}$$

and, for $m \geq 2$,

$$\mathcal{H}_N(U_m) \cong \mathcal{F}\{-2(m-1), 0\} \oplus \mathcal{T}\langle m \rangle \{-m, -m(N+1)\} \| m \| \oplus \bigoplus_{l=1}^{m-1} \mathcal{F}/a\mathcal{F}\langle l \rangle \{-2m+l, -l(N+1)\} \| l+1 \|,$$

where " $\| l \|$ " means shifting the homological grading by l .

1.4. Organization of this paper. In Section 2, we review the definition of \mathcal{H}_N . Then we study the $\mathbb{Q}[a]$ -module structure of \mathcal{H}_N and prove Theorem 1.4 in Section 3. Finally, we prove Theorem 1.6 and Corollary 1.7 in Section 4.

This paper is self-contained for the most part. Of course, some prior knowledge of the Khovanov-Rozansky homology, especially of [4, 9], will be helpful.

2. DEFINITION OF \mathcal{H}_N

In this section, we quickly review the definition of the transverse Khovanov-Rozansky homology \mathcal{H}_N in [9], which is every similar to the definition of the Khovanov-Rozansky homology in [4, 5].

2.1. $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations over $\mathbb{Q}[a, x_1, \dots, x_k]$.

Definition 2.1. We define a $\mathbb{Z}^{\oplus 2}$ -grading on $R = \mathbb{Q}[a, x_1, \dots, x_k]$ by letting $\deg a = (2, 0)$ and $\deg x_i = (0, 2)$ for $i = 1, \dots, k$. We call the first component of this $\mathbb{Z}^{\oplus 2}$ -grading the a -grading and denote its degree function by \deg_a . We call the second component of this $\mathbb{Z}^{\oplus 2}$ -grading the x -grading and denote its degree function by \deg_x . An element of R is said to be homogeneous if it is homogeneous with respect to both the a -grading and the x -grading.

A $\mathbb{Z}^{\oplus 2}$ -graded R -module M is a R -module M equipped with a $\mathbb{Z}^{\oplus 2}$ -grading such that, for any homogeneous element³ m of M , $\deg(am) = \deg m + (2, 0)$ and $\deg(x_i m) = \deg m + (0, 2)$ for $i = 1, \dots, k$. Again, we call the first component of this $\mathbb{Z}^{\oplus 2}$ -grading of M the a -grading and denote its degree function by \deg_a . We call the second component of this $\mathbb{Z}^{\oplus 2}$ -grading of M the x -grading and denote its degree function by \deg_x .

We say that the $\mathbb{Z}^{\oplus 2}$ -grading on M is bounded below if both the a -grading and the x -grading are bounded below.

For a $\mathbb{Z}^{\oplus 2}$ -graded R -module M , we denote by $M\{j, k\}$ the $\mathbb{Z}^{\oplus 2}$ -graded R -module obtained by shifting the $\mathbb{Z}^{\oplus 2}$ -grading of M by (j, k) . That is, for any homogeneous element m of M , $\deg_{M\{j, k\}} m = \deg_M m + (j, k)$.

Definition 2.2. Let w be a homogeneous element with bidegree $(2, 2N + 2)$ of $R = \mathbb{Q}[a, x_1, \dots, x_k]$. A $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization M of w over R is a collection of two $\mathbb{Z}^{\oplus 2}$ -graded free R -modules M_0, M_1 and two homogeneous R -module maps $d_0 : M_0 \rightarrow M_1$, $d_1 : M_1 \rightarrow M_0$ of bidegree $(1, N + 1)$, called differential maps, such that

$$d_1 \circ d_0 = w \cdot \text{id}_{M_0}, \quad d_0 \circ d_1 = w \cdot \text{id}_{M_1}.$$

The \mathbb{Z}_2 -grading of M takes value ε on M_ε . The a - and x -gradings of M are the a - and x -gradings of the underlying $\mathbb{Z}^{\oplus 2}$ -graded R -module $M_0 \oplus M_1$.

We usually write M as $M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0$.

Following [4], we denote by $M\langle 1 \rangle$ the matrix factorization $M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1$ and write $M\langle \underbrace{1 \cdots 1}_{j \text{ times}} \rangle = M\langle 1 \rangle \cdots \langle 1 \rangle$.

For any $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization M of w over R and $j, k \in \mathbb{Z}$, $M\{j, k\}$ is naturally a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization of w over R .

For any two $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations M and M' of w over R , $M \oplus M'$ is naturally a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization of w over R .

Let w and w' be two homogeneous elements of R with bidegree $(2, 2N + 2)$. For $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations M of w and M' of w' over R , the tensor product $M \otimes_R M'$ is the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization of $w + w'$ over R such that:

- $(M \otimes M')_0 = (M_0 \otimes M'_0) \oplus (M_1 \otimes M'_1)$, $(M \otimes M')_1 = (M_1 \otimes M'_0) \oplus (M_0 \otimes M'_1)$;
- The differential is given by the signed Leibniz rule. That is, $d(m \otimes m') = (dm) \otimes m' + (-1)^\varepsilon m \otimes (dm')$ for $m \in M_\varepsilon$ and $m' \in M'$.

Definition 2.3. Let w be a homogeneous element of R with bidegree $(2, 2N + 2)$, and M, M' any two $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations of w over R .

- (1) A morphism of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations from M to M' is a homogeneous R -module homomorphism $f : M \rightarrow M'$ preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -grading satisfying $d_{M'} f = f d_M$. We denote by $\text{Hom}_{\text{mf}}(M, M')$ the \mathbb{Q} -space of all morphisms of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations from M to M' .
- (2) An isomorphism of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations from M to M' is a morphism of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations that is also an isomorphism of the underlying R -modules. We say that M and M' are isomorphic, or $M \cong M'$, if there is an isomorphism from M to M' .
- (3) Two morphisms f and g of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations from M to M' are called homotopic if there is an R -module homomorphism $h : M \rightarrow M'$ shifting the \mathbb{Z}_2 -grading by 1 such that $f - g = d_M h + h d_M$. In this case, we write $f \simeq g$. We denote by $\text{Hom}_{\text{hmf}}(M, M')$ the \mathbb{Q} -space of all homotopy classes of morphisms of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations from M to M' . That is, $\text{Hom}_{\text{hmf}}(M, M') = \text{Hom}_{\text{mf}}(M, M') / \simeq$.

³An element of M is said to be homogeneous if it is homogeneous with respect to both \mathbb{Z} -gradings.

- (4) M and M' are called homotopic, or $M \simeq M'$, if there are morphisms $f : M \rightarrow M'$ and $g : M' \rightarrow M$ such that $g \circ f \simeq \text{id}_M$ and $f \circ g \simeq \text{id}_{M'}$. f and g are called homotopy equivalences between M and M' .
- (5) We say that M is homotopically finite if it is homotopic to a finitely generated graded matrix factorization of w over R .

We define categories $\text{mf}_{R,w}^{\text{all}}$, $\text{mf}_{R,w}$, $\text{hmf}_{R,w}^{\text{all}}$ and $\text{hmf}_{R,w}$ by the following table.

Category	Objects	Morphisms
$\text{mf}_{R,w}^{\text{all}}$	all $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations of w over R with the $\mathbb{Z}^{\oplus 2}$ -grading bounded below	Hom_{mf}
$\text{mf}_{R,w}$	all homotopically finite $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations of w over R with the $\mathbb{Z}^{\oplus 2}$ -grading bounded below	Hom_{mf}
$\text{hmf}_{R,w}^{\text{all}}$	all $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations of w over R with the $\mathbb{Z}^{\oplus 2}$ -grading bounded below	Hom_{hmf}
$\text{hmf}_{R,w}$	all homotopically finite $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations of w over R with the $\mathbb{Z}^{\oplus 2}$ -grading bounded below	Hom_{hmf}

Definition 2.4. If $a_0, a_1 \in R$ are homogeneous elements with $\deg a_0 + \deg a_1 = (2, 2N + 2)$, then denote by $(a_0, a_1)_R$ the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization $R \xrightarrow{a_0} R\{1 - \deg_a a_0, N + 1 - \deg_x a_0\} \xrightarrow{a_1} R$ of $a_0 a_1$ over R . More generally, if $a_{1,0}, a_{1,1}, \dots, a_{l,0}, a_{l,1} \in R$ are homogeneous with $\deg a_{j,0} + \deg a_{j,1} = (2, 2N + 2)$, then denote by

$$\begin{pmatrix} a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} \\ \dots & \dots \\ a_{l,0} & a_{l,1} \end{pmatrix}_R$$

the tensor product $(a_{1,0}, a_{1,1})_R \otimes_R (a_{2,0}, a_{2,1})_R \otimes_R \dots \otimes_R (a_{l,0}, a_{l,1})_R$, which is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization of $\sum_{j=1}^l a_{j,0} a_{j,1}$ over R , and is call the Koszul matrix factorization associated to the above matrix. We drop “ R ” from the notation when it is clear from the context.

Note that the above Koszul matrix factorization is finitely generated over R .

The following proposition from [4] is useful in computing the homology of some MOY graphs.

Proposition 2.5. [4, Proposition 10] *Let I be an ideal of R generated by homogeneous elements. Assume w , a_0 and a_1 are homogeneous elements of R such that $\deg w = \deg a_0 + \deg a_1 = (2, 2N + 2)$ and $w + a_0 a_1 \in I$. Then $w \in I + (a_0)$ and $w \in I + (a_1)$.*

Let M be a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization of w over R , and $\widetilde{M} = M \otimes_R (a_0, a_1)_R$. Then $\widetilde{M}/I\widetilde{M}$, $M/(I + (a_0))M$ and $M/(I + (a_1))M$ are all $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded chain complexes of R -modules.

- (1) *If a_0 is not a zero-divisor in R/I , then there is an R -linear quasi-isomorphism $f : \widetilde{M}/I\widetilde{M} \rightarrow (M/(I + (a_0))M) \langle 1 - \deg_a a_0, N + 1 - \deg_x a_0 \rangle$ that preserves the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -grading.*
- (2) *If a_1 is not a zero-divisor in R/I , then there is an R -linear quasi-isomorphism $g : \widetilde{M}/I\widetilde{M} \rightarrow M/(I + (a_1))M$ that preserves the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -grading.*

2.2. The matrix factorization associated to a MOY graph.

Definition 2.6. A MOY graph Γ is an oriented graph embedded in the plane satisfying:

- (1) Every edge of Γ is colored by 1 or 2.
- (2) Every vertex of Γ is 1-, 2- or 3-valent.
- (3) Every 1-valent vertex of Γ is either the initial point of a 1-colored edge or the terminal point of a 1-colored edge. We call 1-valent vertices of Γ endpoints of Γ .
- (4) Every 2-valent vertex of Γ is the initial point of a 1-colored edge and the terminal point of a 1-colored edge.
- (5) Every 3-valent vertex of Γ is
 - either the initial point of two 1-colored edges and the terminal point of a 2-colored edge,
 - or the terminal point of two 1-colored edges and the initial point of a 2-colored edge.

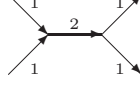


FIGURE 1.

In particular, Definition 2.6 means that every 2-colored edge of Γ has a neighborhood that looks like the local configuration in Figure 1.

Definition 2.7. Let Γ be a MOY graph. A marking of Γ consists of:

- (1) A finite collection of marked points on Γ such that
 - all endpoints are marked,
 - none of the 2- or 3-valent vertices are marked,
 - every 1-colored edge contains a marked point⁴,
 - none of the 2-colored edges contain marked points.
- (2) An assignment that assigns to each marked point a single variable such that no two marked points are assigned the same variable.

Now suppose Γ is a MOY graph with a marking. Let x_1, \dots, x_m be all the variables assigned to marked points on Γ and x_{i_1}, \dots, x_{i_n} all the variables assigned to 1-valent vertices of Γ . We define R to be the $\mathbb{Z}^{\oplus 2}$ -graded ring $R = \mathbb{Q}[a, x_1, \dots, x_m]$ with the $\mathbb{Z}^{\oplus 2}$ -grading given by $\deg a = (2, 0)$ and $\deg x_i = (0, 2)$. Denote by R_{∂} the $\mathbb{Z}^{\oplus 2}$ -graded sub-ring $R_{\partial} = \mathbb{Q}[a, x_{i_1}, \dots, x_{i_n}]$ of R . we call R_{∂} the boundary ring of the marked MOY graph Γ .

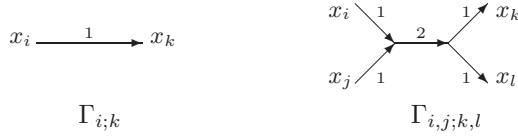


FIGURE 2.

Next, cut Γ at all of its marked points. This breaks Γ into simple marked MOY graphs $\Gamma_1, \dots, \Gamma_p$, each of which is of one of the two types in Figure 2. Note that each Γ_q is marked only at its endpoints. Denote by R_q the $\mathbb{Z}^{\oplus 2}$ -graded polynomial ring over \mathbb{Q} generated by a and the variables marking Γ_q .

- If $\Gamma_q = \Gamma_{i;k}$ in Figure 2, then $R_q = \mathbb{Q}[a, x_i, x_k]$ and

$$(2.1) \quad \mathcal{C}_N(\Gamma_q) = (a \cdot \frac{x_k^{N+1} - x_i^{N+1}}{x_k - x_i}, x_k - x_i)_{R_q}.$$

- If $\Gamma_q = \Gamma_{i,j;k,l}$ in Figure 2, then $R_q = \mathbb{Q}[a, x_i, x_j, x_k, x_l]$ and

$$(2.2) \quad \mathcal{C}_N(\Gamma_q) = \left(a \cdot \frac{g(x_k + x_l, x_k x_l) - g(x_i + x_j, x_k x_l)}{x_k + x_l - x_i - x_j}, \quad x_k + x_l - x_i - x_j \right)_{R_q} \{0, -1\},$$

$$a \cdot \frac{g(x_i + x_j, x_k x_l) - g(x_i + x_j, x_i x_j)}{x_k x_l - x_i x_j}, \quad x_k x_l - x_i x_j$$

where g is the unique 2-variable polynomial satisfying $g(x + y, xy) = x^{N+1} + y^{N+1}$.

Definition 2.8.

$$\mathcal{C}_N(\Gamma) = \bigotimes_{q=1}^p (\mathcal{C}_N(\Gamma_q) \otimes_{R_q} R),$$

where the big tensor product “ $\bigotimes_{q=1}^p$ ” is taken over the ring $R = \mathbb{Q}[a, x_1, \dots, x_m]$.

Note that $\mathcal{C}_N(\Gamma)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization of $w = \sum_{k=1}^n \pm a x_{i_k}^{N+1}$, where the sign is positive if Γ points outward at the corresponding endpoint and negative if Γ points inward at the corresponding endpoint.

⁴We consider the initial and terminal points of an edge part of that edge.

We view $\mathcal{C}_N(\Gamma)$ as an object of the category $\text{hmf}_{R_\partial, w}^{\text{all}}$.

Definition 2.9. A MOY graph is called closed if it has no endpoints. If Γ is a closed MOY graph, then $\mathcal{C}_N(\Gamma)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorization of 0. So it is a homologically \mathbb{Z}_2 -graded chain complex of $\mathbb{Z}^{\oplus 2}$ -graded $\mathbb{Q}[a]$ -modules with a homogeneous differential map. We denote by $\mathcal{H}_N(\Gamma)$ the homology of this chain complex. Note that $\mathcal{H}_N(\Gamma)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded $\mathbb{Q}[a]$ -module by inheriting the gradings of $\mathcal{C}_N(\Gamma)$.

The following two lemmas are slight generalizations of the corresponding results in [4, 5].

Lemma 2.10. [9, Corollary 5.6, Lemma 3.11 and Proposition 7.1] *As matrix factorizations over the respective boundary rings, we have:*

$$(2.3) \quad \mathcal{C}_N \left(\begin{array}{c} \text{loop} \\ \uparrow 1 \quad \uparrow 1 \end{array} \right) \simeq \mathcal{C}_N \left(\begin{array}{c} \text{loop} \\ \uparrow 1 \quad \uparrow 2 \end{array} \right) \{0, 1\} \oplus \mathcal{C}_N \left(\begin{array}{c} \uparrow \\ \uparrow 1 \end{array} \right) \langle 1 \rangle \{-1, 1 - N\},$$

$$(2.4) \quad \mathcal{C}_N \left(\begin{array}{c} \text{loop} \\ \uparrow 1 \quad \uparrow 1 \\ \downarrow 1 \quad \downarrow 1 \end{array} \right) \simeq \mathcal{C}_N \left(\begin{array}{c} \uparrow \\ \uparrow 2 \\ \downarrow 1 \end{array} \right) \{0, -1\} \oplus \mathcal{C}_N \left(\begin{array}{c} \uparrow \\ \uparrow 2 \\ \downarrow 1 \end{array} \right) \{0, 1\},$$

$$(2.5) \quad \mathcal{C}_N \left(\begin{array}{c} \text{complex graph} \\ \uparrow 1 \quad \uparrow 1 \quad \uparrow 1 \\ \downarrow 1 \quad \downarrow 1 \quad \downarrow 1 \end{array} \right) \oplus \mathcal{C}_N \left(\begin{array}{c} \uparrow \\ \uparrow 1 \end{array} \right) \simeq \mathcal{C}_N \left(\begin{array}{c} \text{complex graph} \\ \uparrow 1 \quad \uparrow 1 \quad \uparrow 1 \\ \downarrow 1 \quad \downarrow 1 \quad \downarrow 1 \end{array} \right) \oplus \mathcal{C}_N \left(\begin{array}{c} \uparrow \\ \uparrow 2 \end{array} \right).$$

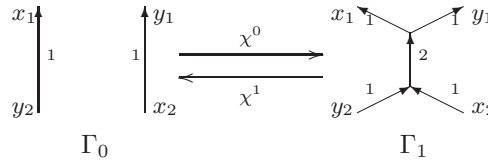


FIGURE 3.

Lemma 2.11. [9, Lemma 3.15] *Let Γ_0 and Γ_1 be the marked MOY graphs in Figure 3. Then there exist morphisms of $\mathbb{Z} \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations $\mathcal{C}_N(\Gamma_0) \xrightarrow{\chi^0} \mathcal{C}_N(\Gamma_1)\{0, -1\}$ and $\mathcal{C}_N(\Gamma_1) \xrightarrow{\chi^1} \mathcal{C}_N(\Gamma_0)\{0, -1\}$ satisfying:*

- (1) χ^0 and χ^1 are homotopically non-trivial,
- (2) $\chi^1 \circ \chi^0 \simeq (x_2 - x_1)\text{id}_{\mathcal{C}_N(\Gamma_0)}$ and $\chi^0 \circ \chi^1 \simeq (x_2 - x_1)\text{id}_{\mathcal{C}_N(\Gamma_1)}$.

Moreover, up to homotopy and scaling,

- χ^0 is the unique homotopically non-trivial morphism of $\mathbb{Z} \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations from $\mathcal{C}_N(\Gamma_0)$ to $\mathcal{C}_N(\Gamma_1)\{0, -1\}$,
- χ^1 is the unique homotopically non-trivial morphism of $\mathbb{Z} \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations from $\mathcal{C}_N(\Gamma_1)$ to $\mathcal{C}_N(\Gamma_0)\{0, -1\}$.

2.3. Definition of \mathcal{H}_N . We first define the chain complex associated to a tangle diagram.

Definition 2.12. Let T be an oriented tangle diagram. We call a segment of T between two adjacent crossings/end points an arc. We color all arcs of T by 1. A marking of T consists of:

- (1) a collections of marked points on T such that
 - none of the crossings of T are marked,
 - all end points are marked,
 - every arc of T contains at least one marked point,
- (2) an assignment of pairwise distinct homogeneous variables of bidegree $(0, 2)$ to the marked points such that every marked point is assigned a unique variable.

Let T be an oriented tangle with a marking. Recall that a is homogeneous of bidegree $(2, 0)$. Denote by

- R the polynomial ring over \mathbb{Q} generated by a and all the variables associated to marked points of T ,
- R_∂ the polynomial ring over \mathbb{Q} generated by a and all the variables associated to end points of T .

Again, we call R_∂ the boundary ring of T .

Cut T at all of its marked points. This cuts T into a collection $\{T_1, \dots, T_l\}$ of simple tangles, each of which is of one of the three types in Figure 4 and is marked only at its end points. Denote by R_i the polynomial ring over \mathbb{Q} generated by a and the variables marking end points of T_i .

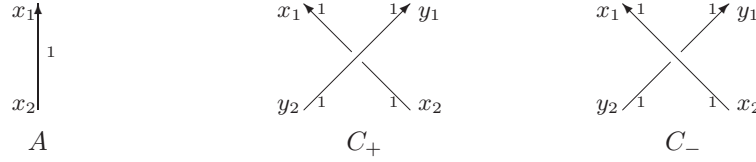


FIGURE 4.

If $T_i = A$, then $R_i = \mathbb{Q}[a, x_1, x_2]$ and $\mathcal{C}_N(T_i)$ is the chain complex over $\text{hmf}_{R_i, a(x_1^{N+1} - x_2^{N+1})}$ given by

$$(2.6) \quad \mathcal{C}_N(T_i) = 0 \rightarrow \underbrace{\mathcal{C}_N(A)}_0 \rightarrow 0,$$

where the $\mathcal{C}_N(A)$ on the right hand side is the matrix factorization associated to the MOY graph A , and the under-brace indicates the homological grading.

If $T_i = C_\pm$, then $R_i = \mathbb{Q}[a, x_1, x_2, y_1, y_2]$ and $\mathcal{C}_N(T_i)$ is the chain complex over $\text{hmf}_{R_i, a(x_1^{N+1} + y_1^{N+1} - x_2^{N+1} - y_2^{N+1})}$ given by

$$(2.7) \quad \mathcal{C}_N(C_+) = 0 \rightarrow \underbrace{\mathcal{C}_N(\Gamma_1) \langle 1 \rangle \{1, N\}}_{-1} \xrightarrow{\chi^1} \underbrace{\mathcal{C}_N(\Gamma_0) \langle 1 \rangle \{1, N-1\}}_0 \rightarrow 0,$$

$$(2.8) \quad \mathcal{C}_N(C_-) = 0 \rightarrow \underbrace{\mathcal{C}_N(\Gamma_0) \langle 1 \rangle \{-1, -N+1\}}_0 \xrightarrow{\chi^0} \underbrace{\mathcal{C}_N(\Gamma_1) \langle 1 \rangle \{-1, -N\}}_1 \rightarrow 0,$$

where Γ_0 and Γ_1 are the resolutions of C_\pm given in Figure 5, the morphisms χ^0 and χ^1 are defined in Lemma 2.11 and the under-braces indicate the homological gradings.

Note that, in all three cases, the differential map of $\mathcal{C}_N(T_i)$ consists of homogeneous morphisms of matrix factorizations preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -grading. Of course, this differential map raises the homological grading by 1.

Definition 2.13. We define the chain complex $\mathcal{C}_N(T)$ associated to T to be

$$\mathcal{C}_N(T) := \bigotimes_{i=1}^l (\mathcal{C}_N(T_i) \otimes_{R_i} R),$$

where the big tensor product “ $\bigotimes_{i=1}^l$ ” is taken over R . We view $\mathcal{C}_N(T)$ as a chain complex of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded matrix factorizations over the ring R_∂ .

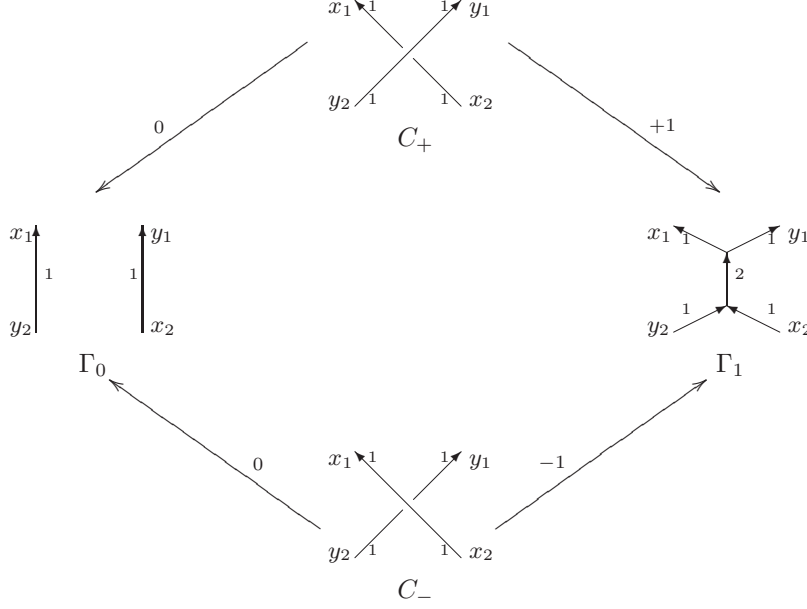


FIGURE 5.

$\mathcal{C}_N(T)$ is equipped with a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -grading, where the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -grading comes from the underlying matrix factorization and the additional \mathbb{Z} -grading is the homological grading.

Note that, if T is an oriented link diagram, then $\mathcal{C}_N(T)$ is a chain complex over the category $\mathbf{hmf}_{\mathbb{Q}[a],0}^{\text{all}}$.

Lemma 2.14. [9, Lemma 4.5, and Propositions 5.5, 6.1, 7.5] *The homotopy type of $\mathcal{C}_N(T)$ is independent of the marking of T and invariant under positive Reidemeister move I and braid-like Reidemeister moves II and III.*

Now let L be a link diagram with a marking. Note $\mathcal{C}_N(L)$ has two differential maps:

- (1) The differential d_{mf} of the underlying matrix factorization structure of $\mathcal{C}_N(L)$.
- (2) The differential d_χ from the crossing information given in equations (2.6), (2.7) and (2.8).

As a matrix factorization, $\mathcal{C}_N(L)$ is a matrix factorization of 0. So $d_{mf}^2 = 0$. Thus, the homology $H(\mathcal{C}_N(L), d_{mf})$ is well defined. In fact, $H(\mathcal{C}_N(L), d_{mf})$ inherits the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -grading of $\mathcal{C}_N(L)$ and $(H(\mathcal{C}_N(L), d_{mf}), d_\chi)$ is a chain complex with a homological \mathbb{Z} -grading of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded $\mathbb{Q}[a]$ -modules.

Definition 2.15. $\mathcal{H}_N(L) := H(H(\mathcal{C}_N(L), d_{mf}), d_\chi)$. It is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -module.

As a simple corollary of Lemma 2.14, we have:

Corollary 2.16. *The $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -module $\mathcal{H}_N(L)$ is independent of the marking of L and invariant under positive Reidemeister move I and braid-like Reidemeister moves II and III.*

Clearly, Theorem 1.1 follows from Lemma 2.14 and Corollary 2.16.

2.4. The $\mathfrak{sl}(N)$ Khovanov-Rozansky homology H_N . If we set $a = 1$ in the above construction, then we get the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology H_N defined in [4]. More precisely, for any tangle T , let

$$(2.9) \quad C_N(T) = \mathcal{C}_N(T)/(a-1)\mathcal{C}_N(T).$$

Then $C_N(T)$ is the $\mathfrak{sl}(N)$ Khovanov-Rozansky chain complex defined in [4]. Note that $C_N(T)$ inherits the \mathbb{Z}_2 -, homological and x -gradings of $\mathcal{C}_N(T)$. It also inherits the differentials d_{mf} and d_χ . For a link diagram L ,

$$(2.10) \quad H_N(L) = H((C_N(L), d_{mf}), d_\chi)$$

is the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology defined in [4]. $H_N(L)$ inherits the gradings of $C_N(L)$ and is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded \mathbb{Q} -linear space.

The $\mathbb{Z}_2 \oplus \mathbb{Z}$ -graded matrix factorization $C_N(\Gamma) = C_N(\Gamma)/(a-1)C_N(\Gamma)$ of a MOY graph Γ satisfies decompositions similar to those in Lemma 2.10.

Lemma 2.17. [4] *As matrix factorizations over the respective boundary rings, we have:*

$$(2.11) \quad C_N \left(\begin{array}{c} \text{loop with 1 on top and 1 on bottom} \end{array} \right) \simeq C_N \left(\begin{array}{c} \text{loop with 1 on top and 2 on bottom} \end{array} \right) \{1\}_x \oplus C_N \left(\begin{array}{c} \text{vertical line with 1 on top and 1 on bottom} \end{array} \right) \langle 1 \rangle \{1-N\}_x,$$

$$(2.12) \quad C_N \left(\begin{array}{c} \text{loop with 2 on top and 2 on bottom} \end{array} \right) \simeq C_N \left(\begin{array}{c} \text{vertical line with 2 on top and 2 on bottom} \end{array} \right) \{-1\}_x \oplus C_N \left(\begin{array}{c} \text{vertical line with 1 on top and 1 on bottom} \end{array} \right) \{1\}_x,$$

$$(2.13) \quad C_N \left(\begin{array}{c} \text{complex graph with multiple crossings and edges labeled 1 and 2} \end{array} \right) \oplus C_N \left(\begin{array}{c} \text{vertical line with 1 on top and 1 on bottom} \end{array} \right) \simeq C_N \left(\begin{array}{c} \text{another complex graph with multiple crossings and edges labeled 1 and 2} \end{array} \right) \oplus C_N \left(\begin{array}{c} \text{vertical line with 1 on top and 1 on bottom} \end{array} \right).$$

In the above, $\{*\}_x$ means shifting the x -grading by $*$.

The following invariance theorem for H_N is established in [4].

Theorem 2.18. [4] *The homotopy type of $C_N(T)$, including its $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -grading, is independent of markings and invariant under all Reidemeister moves. Consequently, every Reidemeister move on L induces an isomorphism of $H_N(L)$ preserving its $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded \mathbb{Q} -linear space structure.*

3. GRADED MODULE STRUCTURE OF \mathcal{H}_N

In this section, we study the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$ -graded $\mathbb{Q}[a]$ -module of \mathcal{H}_N . The goal is to prove Theorem 1.4.

3.1. Resolved braids. In this subsection, we review some basic properties of resolved braids introduced in [8].

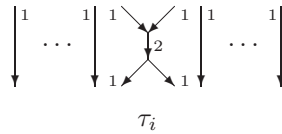


FIGURE 6.

Definition 3.1. For positive integers b, i with $1 \leq i \leq b-1$, let τ_i be the MOY graph depicted in Figure 6. That is, from left to right, τ_i consists of $i-1$ downward 1-colored edges, then a downward 2-colored edge with two 1-colored edges entering through the top and two 1-colored edges exiting through the bottom, and then $b-i-1$ more downward 1-colored edges.

We use $(\tau_{i_1} \cdots \tau_{i_m})_b$ to represent the MOY graph formed by stacking the graphs $\tau_{i_1}, \dots, \tau_{i_m}$ together vertically from top to bottom with the bottom end points of τ_{i_l} identified with the corresponding top end

points of $\tau_{i_{l+1}}$. We call $(\tau_{i_1} \cdots \tau_{i_m})_b$ a resolved braid of b -strands. If the number of strands is clear from the context, then we drop the lower index b and simply write $\tau_{i_1} \cdots \tau_{i_m}$.

Denote by $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ the closed MOY graph obtained from $(\tau_{i_1} \cdots \tau_{i_m})_b$ by attaching a 1-colored edge from each end point at the bottom to the corresponding end point at the top. We call $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ a closed resolved braid of b -strands. Again, if the number of strands is clear from the context, then we drop the lower index b and simply write $\overline{\tau_{i_1} \cdots \tau_{i_m}}$.

We use $(\emptyset)_b$ to represent b vertical downward 1-colored edges, and, therefore, $\overline{(\emptyset)_b}$ represents b concentric 1-colored circles. Again, if the number of strands is clear from the context, then we drop the lower index b .

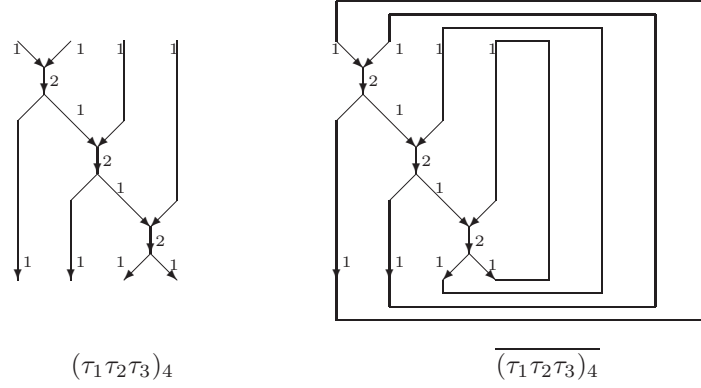


FIGURE 7.

Remark 3.2. (1) Comparing Definition 3.1 to the resolutions in Figure 5, one can see that, if we choose a resolution for every crossing in a (closed) braid, then we get a (closed) resolved braid as defined in Definition 3.1.

(2) There are two obvious types of isotopies of resolved braids and closed resolved braids:

- I_1 : If $|i - j| > 1$, then $\tau_i \tau_j$ is isotopic to $\tau_j \tau_i$;
- I_2 : If μ and ν are two words in $\tau_1, \dots, \tau_{b-1}$, then $\overline{\mu\nu}$ is isotopic to $\overline{\nu\mu}$.

Definition 3.3. We define the weight of the closed resolved braid $\overline{\tau_{i_1} \cdots \tau_{i_m}}$ to be $w(\overline{\tau_{i_1} \cdots \tau_{i_m}}) = i_1 + \cdots + i_m$.

In [8], the author introduced a scheme to perform inductive arguments on the weights of closed resolved braids using the decompositions in Lemma 2.17. The key to this scheme is Corollary 3.5 below, which is a simple consequence of Lemma 3.4.

Lemma 3.4. [8, Lemma 3.5] *Let $\mu = \tau_{i_1} \cdots \tau_{i_m}$ be a resolved braid with b strands satisfying:*

- $m \geq 2$,
- $i_1 = i_m = i$,
- $i_l < i$ for $1 < l < m$.

Then, via a finite sequence of isotopies of type I_1 , μ is isotopic to a resolved braid μ' that contains a segment of the form $\tau_j \tau_j$ or $\tau_j \tau_{j-1} \tau_j$ for some $j \leq i$.

Corollary 3.5. *Let $\overline{\mu}$ be a closed resolved braids with b strands. Then, via a finite sequence of isotopies of types I_1 and I_2 , $\overline{\mu}$ is isotopic to a closed resolved braid of one of the following three types:*

- (a) $\overline{\tau_{i_1} \cdots \tau_{i_m} \tau_i}$, where $i > i_1, \dots, i_m$;
- (b) $\overline{\tau_{i_1} \cdots \tau_{i_m} \tau_j \tau_j}$;
- (c) $\overline{\tau_{i_1} \cdots \tau_{i_m} \tau_j \tau_{j-1} \tau_j}$.

3.2. Homology of closed resolve braids. In this subsection, we study the $\mathbb{Q}[a]$ -module structure of the homology of closed resolved braids. The goal is to establish Lemma 3.9 below.

Lemma 3.6. *Let $(\overline{\emptyset})_b$ be the closed resolved braid with b -strands corresponding to the empty word, that is, the MOY graph consisting of b concentric 1-colored circles. Define the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded $\mathbb{Q}[a]$ -modules \mathcal{M}_0 , \mathcal{M}_1 and \mathcal{M}_∞ by*

$$\begin{aligned}\mathcal{M}_0 &:= \mathbb{Q}[a] \langle 1 \rangle \{-1, 1-N\} \oplus \mathbb{Q}[a], \\ \mathcal{M}_1 &:= \bigoplus_{l=0}^{N-1} \mathbb{Q}[a] \langle 1 \rangle \{-1, 1-N+2l\}, \\ \mathcal{M}_\infty &:= \bigoplus_{l=N}^{\infty} \mathbb{Q}[a]/(a) \langle 1 \rangle \{-1, 1-N+2l\}.\end{aligned}$$

Then, as a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$ -graded $\mathbb{Q}[a]$ -module,

$$\mathcal{H}_N(\overline{(\emptyset)}_b) \cong \mathcal{M}_1^{\otimes b} \oplus \left(\bigoplus_{j=0}^{b-1} \mathcal{M}_0^{\otimes j} \otimes \mathcal{M}_1^{\otimes (b-1-j)} \right) \otimes \mathcal{M}_\infty,$$

where all the tensor products are over $\mathbb{Q}[a]$.

Proof. We prove this lemma by an induction on b . Mark $(\overline{\emptyset})_1$ by a single variable x . Then

$$\mathcal{C}_N(\overline{(\emptyset)}_1) = ((N+1)ax_1^N, 0)_{\mathbb{Q}[a,x]} = \mathbb{Q}[a, x] \xrightarrow{(N+1)ax^N} \mathbb{Q}[a, x] \{-1, 1-N\} \xrightarrow{0} \mathbb{Q}[a, x].$$

So $\mathcal{H}_N(\overline{(\emptyset)}_1) \cong \mathbb{Q}[a, x]/(ax^N) \langle 1 \rangle \{-1, 1-N\} \cong \mathcal{M}_1 \oplus \mathcal{M}_\infty$. This proves the lemma for $b = 1$.

Now assume the lemma is true for $(\overline{\emptyset})_{b-1}$. Consider $(\overline{\emptyset})_b$. Mark the j th circle in $(\overline{\emptyset})_b$ by a single variable x_j . Then

$$\mathcal{C}_N(\overline{(\emptyset)}_b) = \begin{pmatrix} (N+1)ax_1^N & 0 \\ (N+1)ax_2^N & 0 \\ \cdots & \cdots \\ (N+1)ax_b^N & 0 \end{pmatrix}_{\mathbb{Q}[a, x_1, x_2, \dots, x_b]}.$$

Thus, by Proposition 2.5, $\mathcal{C}_N(\overline{(\emptyset)}_b)$ is quasi-isomorphic to

$$\begin{pmatrix} (N+1)ax_1^N & 0 \\ (N+1)ax_2^N & 0 \\ \cdots & \cdots \\ (N+1)ax_{b-1}^N & 0 \end{pmatrix}_{\mathbb{Q}[a, x_1, x_2, \dots, x_{b-1}]} \otimes_{\mathbb{Q}[a]} \mathbb{Q}[a, x_b]/(ax_b^N) \langle 1 \rangle \{-1, 1-N\}.$$

Note that:

- (1) $\mathcal{C}_N(\overline{(\emptyset)}_{b-1}) \cong \begin{pmatrix} (N+1)ax_1^N & 0 \\ (N+1)ax_2^N & 0 \\ \cdots & \cdots \\ (N+1)ax_{b-1}^N & 0 \end{pmatrix}_{\mathbb{Q}[a, x_1, x_2, \dots, x_{b-1}]}$,
- (2) $\mathbb{Q}[a, x_b]/(ax_b^N) \langle 1 \rangle \{-1, 1-N\} \cong \mathcal{M}_1 \oplus \mathcal{M}_\infty$,
- (3) \mathcal{M}_1 is a free $\mathbb{Q}[a]$ -module,
- (4) The homology of $\begin{pmatrix} (N+1)ax_1^N & 0 \\ (N+1)ax_2^N & 0 \\ \cdots & \cdots \\ (N+1)ax_{b-1}^N & 0 \end{pmatrix}_{\mathbb{Q}[a, x_1, x_2, \dots, x_{b-1}]}$ $\otimes_{\mathbb{Q}[a]} \mathcal{M}_\infty$ is isomorphic to $\mathcal{M}_0^{\otimes (b-1)} \otimes \mathcal{M}_\infty$.

Putting the above together, we get

$$\mathcal{H}_N(\overline{(\emptyset)}_b) \cong \mathcal{H}_N(\overline{(\emptyset)}_{b-1}) \otimes_{\mathbb{Q}[a]} \mathcal{M}_1 \oplus \mathcal{M}_0^{\otimes (b-1)} \otimes \mathcal{M}_\infty.$$

This isomorphism and the assumption that the lemma is true for $(\overline{\emptyset})_{b-1}$ imply that the lemma is true for $(\overline{\emptyset})_b$. \square

To discuss the homology of a general closed resolved braid, we need the following lemma, which is a slight refinement of the usual structure theorem of modules over a principal ideal domain.

Lemma 3.7. [9, Lemma 9.2] *Suppose that M is a finitely generated \mathbb{Z} -graded $\mathbb{Q}[a]$ -module. Then, as a \mathbb{Z} -graded $\mathbb{Q}[a]$ -module, $M \cong (\bigoplus_{j=1}^m \mathbb{Q}[a]\{s_j\}_a) \oplus (\bigoplus_{k=1}^n \mathbb{Q}[a]/(a^{l_k})\{t_k\}_a)$, where $\{*\}_a$ means shifting the a -grading by $*$, and the sequences $\{s_1, \dots, s_m\} \subset \mathbb{Z}$, $\{(l_1, t_1), \dots, (l_n, t_n)\} \subset \mathbb{Z}^{\oplus 2}$ are uniquely determined by M up to permutation. We call this decomposition the standard decomposition of M .*

Definition 3.8. For a closed resolved braid $\bar{\mu}$, we denote by $\mathcal{H}_N^{\varepsilon, j, k}(\bar{\mu})$ (resp. $\mathcal{C}_N^{\varepsilon, j, k}(\bar{\mu})$) the homogeneous component of $\mathcal{H}_N(\bar{\mu})$ (resp. $\mathcal{C}_N(\bar{\mu})$) of \mathbb{Z}_2 -degree ε , a -degree j and x -degree k . If we replace one of these indices by a \star , it means we direct sum the components over all possible values of that index. For example, $\mathcal{H}_N^{\varepsilon, \star, k}(\bar{\mu}) = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}_N^{\varepsilon, j, k}(\bar{\mu})$.

Similarly, we denote by $H_N^{\varepsilon, k}(\bar{\mu})$ (resp. $C_N^{\varepsilon, k}(\bar{\mu})$) the homogeneous component of \mathbb{Z}_2 -degree ε and x -degree k of the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology $H_N(\bar{\mu})$ (resp. $C_N(\bar{\mu})$) of $\bar{\mu}$.

Lemma 3.9. *For a closed resolved braid $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ of b strands, we have that, as a \mathbb{Z} -graded $\mathbb{Q}[a]$ -module,*

$$\mathcal{H}_N^{\varepsilon, \star, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}) \cong H_N^{\varepsilon, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{-b\}_a \oplus \bigoplus_{i=1}^l \mathbb{Q}[a]/(a)\{s_i\}_a,$$

where

- we give $H_N^{\varepsilon, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ the a -grading 0, and $\{*\}_a$ means shifting the a -grading by $*$,
- up to permutation, the sequence $\{s_1, \dots, s_l\}$ is uniquely determined by $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$, N , k and ε ,
- $-b \leq s_i \leq -1$ and $(N-1)s_i \leq k - 2N + m$ for $i = 1, \dots, l$.

Proof. From the construction of $\mathcal{C}_N(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$, one can see that $\mathcal{C}_N^{\varepsilon, \star, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ is a finitely generated free $\mathbb{Q}[a]$ -module. This implies that $\mathcal{H}_N^{\varepsilon, \star, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ is a finitely generated \mathbb{Z} -graded $\mathbb{Q}[a]$ -module. So, by Lemma 3.7, $\mathcal{H}_N^{\varepsilon, \star, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ has a unique standard decomposition. Now, to prove the lemma, we only need to verify that:

- (I) The free part of $\mathcal{H}_N^{\varepsilon, \star, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ is isomorphic to $H_N^{\varepsilon, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{-b\}_a$.
- (II) All torsion components of $\mathcal{H}_N^{\varepsilon, \star, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ are of the form $\mathbb{Q}[a]/(a)\{s\}_a$.
- (III) If $\mathcal{H}_N^{\varepsilon, \star, k}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ contains a torsion component $\mathbb{Q}[a]/(a)\{s\}_a$, then $-b \leq s \leq -1$ and $(N-1)s \leq k - 2N + m$.

These three conclusions can be easily proved by an induction on the weight of $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ using Lemmas 2.10, 2.17 and Corollary 3.5.

If the weight of a closed resolved braid is 0, then it is $\overline{(\emptyset)_b}$. By Lemma 3.6, (I-III) is true for $\overline{(\emptyset)_b}$ for all $b \geq 0$.

Now assume that (I-III) is true for all closed resolved braids (on any number of strands) with weight less than the weight of $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$. By Corollary 3.5, via a finite sequence of isotopies of types I_1 and I_2 , $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ is isotopic to a closed resolved braid of one of the following three types:

- (a) $\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}} \tau_i)_b}$, where $i > j_1, \dots, j_m$;
- (b) $\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j \tau_j)_b}$;
- (c) $\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_j \tau_{j-1} \tau_j)_b}$.

Of course, isotopies of types I_1 and I_2 do not change the weight of a closed resolved braid.

In Case (a), we have

$$\begin{aligned} \mathcal{H}_N(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}) &\cong \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}} \tau_i)_b}), \\ \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_b}) &\cong \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}} \tau_i)_b})\{0, 1\} \oplus \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_{b-1}})\{-1, 1-N\}, \\ H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_b}) &\cong H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}} \tau_i)_b})\{1\}_x \oplus H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_{b-1}})\{1-N\}_x, \end{aligned}$$

where the second and third isomorphisms follow from Lemmas 2.10 and 2.17. The weights of both $\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_b}$ and $\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_{b-1}}$ are less than that of $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$. So (I-III) are true for $\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_b}$ and $\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_{b-1}}$.

Moreover, by Lemma 3.7, the standard decomposition of $\mathcal{H}_N^{\varepsilon, \star, k}(\overline{(\tau_{j_1} \cdots \tau_{j_{m-1}})_b})$ is unique. It then follows from the above isomorphisms that (I-III) are true for $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ too.

In Case (b), we have

$$\begin{aligned}\mathcal{H}_N(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}) &\cong \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j \tau_j)_b}), \\ \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j \tau_j)_b}) &\cong \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j)_b})\{0, 1\} \oplus \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j)_b})\{0, -1\}, \\ H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j \tau_j)_b}) &\cong H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j)_b})\{1\}_x \oplus H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j)_b})\{-1\}_x,\end{aligned}$$

where the second and third isomorphisms follow from Lemmas 2.10 and 2.17. The weight of $\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j)_b}$ is less than that of $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$. So (I-III) are true for $\overline{(\tau_{j_1} \cdots \tau_{j_{m-2}} \tau_j)_b}$. It then follows from the above isomorphisms that (I-III) are true for $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ too.

In Case (c), we have

$$\begin{aligned}\mathcal{H}_N(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}) &\cong \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_j \tau_{j-1} \tau_j)_b}), \\ \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_j \tau_{j-1} \tau_j)_b}) \oplus \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_{j-1})_b}) &\cong \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_{j-1} \tau_j \tau_{j-1})_b}) \oplus \mathcal{H}_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_j)_b}), \\ H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_j \tau_{j-1} \tau_j)_b}) \oplus H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_{j-1})_b}) &\cong H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_{j-1} \tau_j \tau_{j-1})_b}) \oplus H_N(\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_j)_b}),\end{aligned}$$

where the second and third isomorphisms follow from Lemmas 2.10 and 2.17. The weights of $\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_{j-1})_b}$, $\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_{j-1} \tau_j \tau_{j-1})_b}$ and $\overline{(\tau_{j_1} \cdots \tau_{j_{m-3}} \tau_j)_b}$ are less than that of $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$. So (I-III) are true for these three resolved closed braids. It then follows from the above isomorphisms that (I-III) are true for $\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}$ too. \square

Corollary 3.10. $\mathcal{H}^{b+1, \star, \star}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b})$ is a direct sum of components of the form $\mathbb{Q}[a]/(a)\{s, k\}$

Proof. From [4], we know that $H^{b+1, \star}(\overline{(\tau_{i_1} \cdots \tau_{i_m})_b}) \cong 0$. So the corollary follows from Lemma 3.9. \square

3.3. Homology of a closed braid. We are now ready to prove Theorem 1.4.

Let B be a closed braid of b strands. Recall that $\mathcal{H}_N(B) = H(H(\mathcal{C}_N(B), d_{mf}), d_\chi)$. Denote by $H^{\varepsilon, i, j, k}(\mathcal{C}_N(B), d_{mf})$ the homogeneous component of $H(\mathcal{C}_N(B), d_{mf})$ of \mathbb{Z}_2 -degree ε , homological degree i , a -degree j and x -degree k . We use the \star -notation as introduced in Definition 1.3. Then, for every $(\varepsilon, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}$, $(H^{\varepsilon, \star, \star, k}(\mathcal{C}_N(B), d_{mf}), d_\chi)$ is a bounded chain complex of finitely generated \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules. Denote by $F^{\varepsilon, i, \star, k}$ the free part of $H^{\varepsilon, i, \star, k}(\mathcal{C}_N(B), d_{mf})$ and by $T^{\varepsilon, i, \star, k}$ the torsion part of $H^{\varepsilon, i, \star, k}(\mathcal{C}_N(B), d_{mf})$. Note that $sl(B) = c_+ - c_- - b$, where c_\pm is the number of \pm crossings in B . Then, by Lemma 3.9,

- $F^{\varepsilon, i, \star, k} \cong H^{\varepsilon, i, k}(\mathcal{C}_N(B), d_{mf}) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$,
- $T^{\varepsilon, i, \star, k}$ is a direct sum of finitely many components of the form $\mathbb{Q}[a]/(a)\{s\}_a$.

Under the decomposition $H^{\varepsilon, i, \star, k}(\mathcal{C}_N(B), d_{mf}) = \begin{matrix} F^{\varepsilon, i, \star, k} \\ \oplus \\ T^{\varepsilon, i, \star, k} \end{matrix}$, then differential map $H^{\varepsilon, i, \star, k}(\mathcal{C}_N(B), d_{mf}) \xrightarrow{d_\chi^i}$

$H^{\varepsilon, i+1, \star, k}(\mathcal{C}_N(B), d_{mf})$ takes the form

$$\begin{matrix} F^{\varepsilon, i, \star, k} \\ \oplus \\ T^{\varepsilon, i, \star, k} \end{matrix} \xrightarrow{\begin{pmatrix} d_{\chi, FF}^i & 0 \\ d_{\chi, FT}^i & d_{\chi, TT}^i \end{pmatrix}} \begin{matrix} F^{\varepsilon, i+1, \star, k} \\ \oplus \\ T^{\varepsilon, i+1, \star, k} \end{matrix},$$

where $d_{\chi, FF}^i$, $d_{\chi, FT}^i$ and $d_{\chi, TT}^i$ are homogeneous homomorphisms of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules preserving the a -grading. This give rise to two chain complexes $(F^{\varepsilon, \star, \star, k}, d_{\chi, FF})$ and $(T^{\varepsilon, \star, \star, k}, d_{\chi, TT})$. Moreover, $(H^{\varepsilon, \star, \star, k}(\mathcal{C}_N(B), d_{mf}), d_\chi)$ is isomorphic to the mapping cone of the chain map $F^{\varepsilon, \star, \star, k}\|1\| \xrightarrow{d_{\chi, FT}} T^{\varepsilon, \star, \star, k}$, where “ $\| \cdot \|$ ” means shifting the homological grading up by \star . Thus, we get the follow lemma.

Lemma 3.11. *There is a short exact sequence*

$$0 \rightarrow T^{\varepsilon, \star, \star, k} \rightarrow \mathcal{C}_N^{\varepsilon, \star, \star, k}(B) \rightarrow F^{\varepsilon, \star, \star, k} \rightarrow 0,$$

which induces a long exact sequence

$$\dots \xrightarrow{d_{\chi, FT}^{i-1}} H^i(T^{\varepsilon, \star, \star, k}) \rightarrow \mathcal{H}_N^{\varepsilon, i, \star, k}(B) \rightarrow H^i(F^{\varepsilon, \star, \star, k}) \xrightarrow{d_{\chi, FT}^i} H^{i+1}(T^{\varepsilon, \star, \star, k}) \rightarrow \dots$$

of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules, where the arrows preserve the $\mathbb{Q}[a]$ -grading.

Proof. This lemma follows from the standard construction of a long exact sequence from a mapping cone. \square

Lemma 3.12. $H^i(F^{\varepsilon, \star, \star, k}) \cong H_N^{\varepsilon, i, k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$ for every $(\varepsilon, i, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$.

Proof. Recall that $C_N(B) := \mathcal{C}_N(B)/(a-1)\mathcal{C}_N(B)$ and $\mathcal{C}_N(B)$ is a free $\mathbb{Q}[a]$ -module. So there is a short exact sequence

$$0 \rightarrow \mathcal{C}_N(B) \xrightarrow{a-1} \mathcal{C}_N(B) \rightarrow C_N(B) \rightarrow 0.$$

This induces a long exact sequence

$$\dots \rightarrow H^{\varepsilon, i, \star, \star}(\mathcal{C}_N(B), d_{mf}) \xrightarrow{a-1} H^{\varepsilon, i, \star, \star}(\mathcal{C}_N(B), d_{mf}) \rightarrow H^{\varepsilon, i, \star, \star}(C_N(B), d_{mf}) \rightarrow H^{\varepsilon+1, i, \star, \star}(C_N(B), d_{mf})\{-1, -N-1\} \xrightarrow{a-1} \dots$$

preserving the x -grading. By [9, Lemma 9.1], the multiplication by $a-1$ is an injective endomorphism of $H^{\varepsilon, i, \star, \star}(\mathcal{C}_N(B), d_{mf})$. So this long exact sequence breaks into a short exact sequence

$$0 \rightarrow (H^{\varepsilon, \star, \star, \star}(\mathcal{C}_N(B), d_{mf}), d_{\chi}) \xrightarrow{a-1} (H^{\varepsilon, \star, \star, \star}(\mathcal{C}_N(B), d_{mf}), d_{\chi}) \rightarrow (H^{\varepsilon, \star, \star, \star}(C_N(B), d_{mf}), d_{\chi}) \rightarrow 0.$$

This shows that the chain complexes $(H(C_N(B), d_{mf}), d_{\chi})$ and $(H(\mathcal{C}_N(B), d_{mf})/(a-1)H(\mathcal{C}_N(B), d_{mf}), d_{\chi})$ are isomorphic to each other, and the isomorphism preserves the \mathbb{Z}_2 -, homological and x -gradings.

From the decomposition $H^{\varepsilon, i, \star, k}(\mathcal{C}_N(B), d_{mf}) = \bigoplus_{T^{\varepsilon, i, \star, k}} \dots$, it is clear that

$$(H^{\varepsilon, \star, \star, k}(\mathcal{C}_N(B), d_{mf})/(a-1)H^{\varepsilon, \star, \star, k}(\mathcal{C}_N(B), d_{mf}), d_{\chi}) \cong (F^{\varepsilon, \star, \star, k}/(a-1)F^{\varepsilon, \star, \star, k}, d_{\chi, FF}).$$

So there is an isomorphism of chain complexes

$$(3.1) \quad (F^{\varepsilon, \star, \star, k}/(a-1)F^{\varepsilon, \star, \star, k}, d_{\chi, FF}) \cong (H^{\varepsilon, \star, k}(C_N(B), d_{mf}), d_{\chi}).$$

Recall that $d_{\chi, FF}$ preserves the a -grading and $F^{\varepsilon, i, \star, k} \cong H^{\varepsilon, i, k}(C_N(B), d_{mf}) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$, where the shift of the a -grading is independent of the homological grading i . Now let $n_i = \dim_{\mathbb{Q}} H^{\varepsilon, i, k}(C_N(B), d_{mf})$ and fix a basis for $H^{\varepsilon, i, k}(C_N(B), d_{mf})$. This basis induces a $\mathbb{Q}[a]$ -basis for $F^{\varepsilon, i, \star, k}$ and allows us to identify $F^{\varepsilon, i, \star, k}$ with $\mathbb{Q}[a]^{\oplus n_i}\{sl(B)\}_a$. Thus, $(F^{\varepsilon, \star, \star, k}, d_{\chi, FF})$ is isomorphic to the chain complex

$$C = \dots \xrightarrow{D_{i-1}} \mathbb{Q}[a]^{\oplus n_i}\{sl(B)\}_a \xrightarrow{D_i} \mathbb{Q}[a]^{\oplus n_{i+1}}\{sl(B)\}_a \xrightarrow{D_{i+1}} \dots,$$

where D_i is the matrix of $d_{\chi, FF}^i$ relative to the bases of $F^{\varepsilon, i, \star, k}$ and $F^{\varepsilon, i+1, \star, k}$. Since $d_{\chi, FF}$ preserves the a -grading, all entries of D_i are elements of \mathbb{Q} . Consider the chain complex

$$\hat{C} = \dots \xrightarrow{D_{i-1}} \mathbb{Q}^{\oplus n_i} \xrightarrow{D_i} \mathbb{Q}^{\oplus n_{i+1}} \xrightarrow{D_{i+1}} \dots$$

One can see that $(F^{\varepsilon, \star, \star, k}, d_{\chi, FF}) \cong C \cong \hat{C} \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$ and, by isomorphism (3.1), $\hat{C} \cong C/(a-1)C \cong (H^{\varepsilon, \star, k}(C_N(B), d_{mf}), d_{\chi})$. Combining these, we get

$$(F^{\varepsilon, \star, \star, k}, d_{\chi, FF}) \cong (H^{\varepsilon, \star, k}(C_N(B), d_{mf}) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a, d_{\chi}).$$

This implies that $H^i(F^{\varepsilon, \star, \star, k}) \cong H_N^{\varepsilon, i, k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$. \square

Proof of Theorem 1.4. From Lemmas 3.11 and 3.12, we get a long exact sequence

$$(3.2) \quad \dots \rightarrow H^i(T^{\varepsilon, \star, \star, k}) \rightarrow \mathcal{H}_N^{\varepsilon, i, \star, k}(B) \rightarrow H_N^{\varepsilon, i, k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a \xrightarrow{d_{\chi, FT}^i} H^{i+1}(T^{\varepsilon, \star, \star, k}) \rightarrow \dots$$

Denote by $F\mathcal{H}_N^{\varepsilon, i, \star, k}(B)$ the free part of the \mathbb{Z} -graded $\mathbb{Q}[a]$ -module $\mathcal{H}_N^{\varepsilon, i, \star, k}(B)$ and by $T\mathcal{H}_N^{\varepsilon, i, \star, k}(B)$ the torsion part of $\mathcal{H}_N^{\varepsilon, i, \star, k}(B)$. Then the long exact sequence (3.2) splits into two exact sequences:

$$(3.3) \quad \dots \rightarrow H^i(T^{\varepsilon, \star, \star, k}) \rightarrow T\mathcal{H}_N^{\varepsilon, i, \star, k}(B) \rightarrow 0,$$

$$(3.4) \quad 0 \rightarrow F\mathcal{H}_N^{\varepsilon, i, \star, k}(B) \xrightarrow{f} H_N^{\varepsilon, i, k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a \xrightarrow{d_{\chi, FT}^i} H^{i+1}(T^{\varepsilon, \star, \star, k}) \rightarrow \dots$$

Since $T^{\varepsilon,i,\star,k}$ is a direct sum of finitely many components of the form $\mathbb{Q}[a]/(a)\{s\}_a$, so is $H^i(T^{\varepsilon,\star,\star,k})$. From the exact sequence (3.3), one can see that $T\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$ is a quotient module of $H^i(T^{\varepsilon,\star,\star,k})$. Thus, $T\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$ is also a direct sum of finitely many components of the form $\mathbb{Q}[a]/(a)\{s\}_a$. That is,

$$(3.5) \quad T\mathcal{H}_N^{\varepsilon,i,\star,k}(B) \cong \left(\bigoplus_{q=1}^n \mathbb{Q}[a]/(a)\{s_q\}_a \right),$$

for some finite sequence $\{s_1, \dots, s_n\}$ of integers.

Next we prove that the \mathbb{Q} -linear map

$$F\mathcal{H}_N^{\varepsilon,i,\star,k}(B)/(a-1)F\mathcal{H}_N^{\varepsilon,i,\star,k}(B) \xrightarrow{f} H_N^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a / (a-1)H_N^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$$

is an isomorphism. First, note that $H^{i+1}(T^{\varepsilon,\star,\star,k})$ is a direct sum of components of the form $\mathbb{Q}[a]/(a)\{s\}_a$. So any multiple of a in $H_N^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$ is in $\ker d_{\chi,FT}^i = \text{Im} f$. For any $u \in F\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$ such that $f(u) = (a-1)v$ for some $v \in H_N^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$, there exists an $u' \in F\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$ satisfying $f(u') = av$. Thus,

$$f(-(a-1)(u-u')) = -(a-1)(f(u) - f(u')) = (a-1)v = f(u).$$

But $F\mathcal{H}_N^{\varepsilon,i,\star,k}(B) \xrightarrow{f} H_N^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$ is injective. So $u = -(a-1)(u-u')$. This shows that the above \mathbb{Q} -linear map is injective. Second, for every $v \in H_N^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$, there is a $u \in F\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$ such that $f(u) = av$. So $v = f(u) - (a-1)v$. This shows that the above \mathbb{Q} -linear map is surjective. Thus, it is an isomorphism.

The above \mathbb{Q} -linear isomorphism implies that the rank of the \mathbb{Z} -graded free $\mathbb{Q}[a]$ -module $F\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$ is equal to $\dim_{\mathbb{Q}} H_N^{\varepsilon,i,k}(B)$. Hence, by Lemma 3.7,

$$(3.6) \quad F\mathcal{H}_N^{\varepsilon,i,\star,k}(B) \cong \bigoplus_{p=1}^{\dim_{\mathbb{Q}} H_N^{\varepsilon,i,k}(B)} \mathbb{Q}[a]\{t_p\}_a.$$

From [4], we know that $H_N^{sl(B)-1,i,k}(B) \cong 0$ for any i, k . So, for any i, k ,

$$(3.7) \quad F\mathcal{H}_N^{sl(B)-1,i,\star,k}(B) \cong 0.$$

From the construction of $\mathcal{H}_N(B)$, one can see that, when $\varepsilon = sl(B)$, the parity of t_p in (3.6) must be the same as that of $sl(B)$. Since $F\mathcal{H}_N^{sl(B),i,\star,k}(B) \xrightarrow{f} H_N^{sl(B),i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$ is injective and preserves the a -grading, we know that $t_p \geq sl(B)$ if $\varepsilon = sl(B)$. Assume that $F\mathcal{H}_N^{sl(B),i,\star,k}(B)$ contains a component $\mathbb{Q}[a]\{t_p\}_a$ such that $t_p \geq sl(B) + 4$. Denote by 1_p the 1 in $\mathbb{Q}[a]\{t_p\}_a$. Then $f(1_p) = a^2v$ for some $v \in H_N^{sl(B),i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{sl(B)\}_a$. Consider the exact sequence (3.4). Again, since $H^{i+1}(T^{sl(B),\star,\star,k})$ is a direct sum of finitely many components of the form $\mathbb{Q}[a]/(a)\{s\}_a$, one can see that $av \in \ker d_{\chi,FT}^i = \text{Im} f$. So there exists a $u \in F\mathcal{H}_N^{sl(B),i,\star,k}(B)$ such that $f(u) = av$. Therefore, $f(1_p) = f(au)$. But f is injective. This means $1_p = au$, which is a contradiction. Thus, when $\varepsilon = sl(B)$, we have $t_p = sl(B)$ or $sl(B) + 2$ for every p and

$$(3.8) \quad F\mathcal{H}_N^{sl(B),i,\star,k}(B) \cong (\mathbb{Q}[a]\{sl(B)\}_a)^{\oplus l} \oplus (\mathbb{Q}[a]\{sl(B) + 2\}_a)^{\oplus (\dim_{\mathbb{Q}} H_N^{sl(B),i,k}(B) - l)}$$

for some non-negative integer l .

By decompositions (3.5), (3.7) and (3.8), one can see that $\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$ admits a decomposition of the form given in Theorem 1.4. The uniqueness of this decomposition follows from Lemma 3.7. The only things left to prove are the bounds for s_q . In the remainder of this proof, we show that the bound for s_q in Theorem 1.4 follow from the corresponding bounds in Lemma 3.9.

If we choose a resolution as in Figure 5 for each crossing of B , we get a closed resolved braid. We call such a closed resolved braid a resolution of B and denote by $\mathcal{R}(B)$ the set of all resolutions of B . As suggested in Figure 5, we call the resolution $C_{\pm} \rightsquigarrow \Gamma_0$ a 0-resolution and $C_{\pm} \rightsquigarrow \Gamma_1$ a ± 1 -resolution. For $\bar{\mu} \in \mathcal{R}(B)$, assume it contains $m_{\bar{\mu},+} + m_{\bar{\mu},-}$ 2-colored edges, where $m_{\bar{\mu},\pm}$ is the number of 2-colored edges in $\bar{\mu}$ coming

from ± 1 -resolutions. From the construction of $\mathcal{C}_N(B)$, especially local chain complexes (2.7) and (2.8), one can see that

$$(3.9) \quad \mathcal{C}_N(B) = \bigoplus_{\overline{\mu} \in \mathcal{R}(B)} \mathcal{C}_N(\overline{\mu}) \langle w \rangle \{w, (N-1)w + m_{\overline{\mu},+} - m_{\overline{\mu},-}\} \|m_{\overline{\mu},-} - m_{\overline{\mu},+}\|,$$

where $w = c_+ - c_-$ is the writhe of B and “ $\| * \|$ ” means shifting the homological grading by $*$. From Lemma 3.9, we know that, if $\mathcal{H}_N^{\varepsilon,*,k-(N-1)w-m_{\overline{\mu},+}+m_{\overline{\mu},-}}(\overline{\mu})\{w\}_a$ contains a torsion component $\mathbb{Q}[a]/(a)\{s\}_a$, then $w-b \leq s \leq w-1$ and $(N-1)s \leq k-2N+2m_{\overline{\mu},-}$. Note that $w-b = sl(B)$ and $m_{\overline{\mu},-} \leq c_-$. So, by decomposition (3.9), we have that, if $T^{\varepsilon,*,*,k}$ contains a component $\mathbb{Q}[a]/(a)\{s\}_a$, then

$$(3.10) \quad sl(B) \leq s \leq w-1 \text{ and } (N-1)s \leq k-2N+2c_-.$$

Therefore, if $H^i(T^{\varepsilon,*,*,k})$ contains a component $\mathbb{Q}[a]/(a)\{s\}_a$, then s satisfies the two bounds in (3.10). Finally, by the exact sequence (3.3), $T\mathcal{H}_N^{\varepsilon,i,*,k}(B)$ is a quotient module of $H^i(T^{\varepsilon,*,*,k})$. So, if $\mathcal{H}_N^{\varepsilon,i,*,k}(B)$ contains a component $\mathbb{Q}[a]/(a)\{s\}_a$, then s satisfies the two bounds in (3.10). This completes the proof of Theorem 1.4. \square

4. STABILIZATION

In this section, we study how \mathcal{H}_N changes under stabilization. The goal is to prove Theorem 1.6.

4.1. Mapping cones. We now review some basic properties of mapping cones.

Definition 4.1. Let A, B be two chain complexes of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules and $f : A \rightarrow B$ a chain map preserving both the homological grading and the a -grading. Then the mapping cone $\text{cone}(f)$ is defined to be the chain complex given by:

- $\text{cone}^i(f) = \begin{matrix} A^i \\ \oplus \\ B^{i-1} \end{matrix}$,
- the differential $\text{cone}^i(f) \xrightarrow{d} \text{cone}^{i+1}(f)$ is the map $\begin{matrix} A^i \\ \oplus \\ B^{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} d_A & 0 \\ f & d_B \end{pmatrix}} \begin{matrix} A^{i+1} \\ \oplus \\ B^i \end{matrix}$, where d_A and d_B are the differential maps of A and B .

Lemma 4.2. Suppose that $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short exact sequence of chain complexes of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules, where f and g preserve both the homological grading and the a -grading. Then, as \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules, $H^i(\text{cone}(f)) \cong H^{i-1}(C)$ and $H^i(\text{cone}(g)) \cong H^i(A)$.

Proof. Denote by id_A the identity map from A to itself. Define $\alpha : \text{cone}(\text{id}_A) \rightarrow \text{cone}(f)$ by $\begin{matrix} A^i \\ \oplus \\ A^{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \text{id}_A & 0 \\ 0 & f \end{pmatrix}}$ and $\beta : \text{cone}(f) \rightarrow C[1]$ by $\begin{matrix} A^i \\ \oplus \\ B^{i-1} \end{matrix} \xrightarrow{(0,g)} C^{i-1}$. Then α, β are chain maps and

$$0 \rightarrow \text{cone}(\text{id}_A) \xrightarrow{\alpha} \text{cone}(f) \xrightarrow{\beta} C[1] \rightarrow 0$$

is a short exact sequence. It induces a long exact sequence

$$\cdots \rightarrow H^i(\text{cone}(\text{id}_A)) \rightarrow H^i(\text{cone}(f)) \rightarrow H^{i-1}(C) \rightarrow H^{i+1}(\text{cone}(\text{id}_A)) \rightarrow \cdots$$

Since $H(\text{cone}(\text{id}_A)) \cong 0$. This long exact sequence implies that $H^i(\text{cone}(f)) \cong H^{i-1}(C)$.

Now define $\phi : A \rightarrow \text{cone}(g)$ by $A^i \xrightarrow{\begin{pmatrix} f \\ 0 \end{pmatrix}} \begin{matrix} B^i \\ \oplus \\ C^{i-1} \end{matrix}$ and $\psi : \text{cone}(g) \rightarrow \text{cone}(\text{id}_C)$ by $\begin{matrix} B^i \\ \oplus \\ C^{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & \text{id}_C \end{pmatrix}}$
 $\begin{matrix} C^i \\ \oplus \\ C^{i-1} \end{matrix}$. Then ϕ, ψ are chain maps and

$$0 \rightarrow A \xrightarrow{\phi} \text{cone}(g) \xrightarrow{\psi} \text{cone}(\text{id}_C) \rightarrow 0$$

is a short exact sequence. It induces a long exact sequence

$$\cdots \rightarrow H^{i-1}(\text{cone}(\text{id}_C)) \rightarrow H^i(A) \rightarrow H^i(\text{cone}(g)) \rightarrow H^i(\text{cone}(\text{id}_C)) \rightarrow \cdots$$

Since $H(\text{cone}(\text{id}_C)) \cong 0$, this long exact sequence implies that $H^i(\text{cone}(g)) \cong H^i(A)$. \square

Lemma 4.3. Suppose that $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \rightarrow 0$ is an exact sequence of chain complexes of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules, where f, g and h preserve both the homological grading and the a -grading. Then there is a long exact sequence of \mathbb{Z} -graded $\mathbb{Q}[a]$ -modules

$$\cdots \rightarrow H^i(A) \rightarrow H^i(\text{cone}(g)) \rightarrow H^{i-1}(D) \rightarrow H^{i+1}(A) \rightarrow \cdots$$

Proof. Denote by $\pi : B \rightarrow B/f(A)$ the standard quotient map. Define $\alpha : \text{cone}(\pi) \rightarrow \text{cone}(g)$ by

$$\begin{matrix} B^i \\ \oplus \\ B^{i-1}/f(A^{i-1}) \end{matrix} \xrightarrow{\begin{pmatrix} \text{id}_B & 0 \\ 0 & g \end{pmatrix}} \begin{matrix} B^i \\ \oplus \\ C^{i-1} \end{matrix}, \text{ which is well defined since } \ker g = \text{Im } f. \text{ Also, define } \beta : \text{cone}(g) \rightarrow$$

$$D[1] \text{ by } \begin{matrix} B^i \\ \oplus \\ C^{i-1} \end{matrix} \xrightarrow{(0,h)} D^{i-1}. \text{ Then } \alpha, \beta \text{ are chain maps and}$$

$$0 \rightarrow \text{cone}(\pi) \xrightarrow{\alpha} \text{cone}(g) \xrightarrow{\beta} D[1] \rightarrow 0$$

is a short exact sequence. It induces a long exact sequence

$$\cdots \rightarrow H^i(\text{cone}(\pi)) \rightarrow H^i(\text{cone}(g)) \rightarrow H^{i-1}(D) \rightarrow H^{i+1}(\text{cone}(\pi)) \rightarrow \cdots$$

But $0 \rightarrow A \xrightarrow{f} B \xrightarrow{\pi} B/f(A) \rightarrow 0$ is a short exact sequence of complexes. So, by Lemma 4.2, we know that $H^i(\text{cone}(\pi)) \cong H^i(A)$. Thus, we have a long exact sequence

$$\cdots \rightarrow H^i(A) \rightarrow H^i(\text{cone}(g)) \rightarrow H^{i-1}(D) \rightarrow H^{i+1}(A) \rightarrow \cdots$$

\square

4.2. Stabilization and \mathcal{H}_N . Next, we prove Theorem 1.6.

Proof of Theorem 1.6. Let B be a closed braid. Set $\mathcal{C}_N(B) = \mathcal{C}_N(B)/a\mathcal{C}_N(B)$. Recall that π_0 is the standard quotient map $\mathcal{C}_N(B) \xrightarrow{\pi_0} \mathcal{C}_N(B)/a\mathcal{C}_N(B) = \mathcal{C}_N(B)$. Then there is a short exact sequence

$$0 \rightarrow \mathcal{C}_N(B) \xrightarrow{a} \mathcal{C}_N(B)\{-2, 0\} \xrightarrow{\pi_0} \mathcal{C}_N(B)\{-2, 0\} \rightarrow 0.$$

Note that d_{mf} is homogeneous with \mathbb{Z}_2 -degree 1, homological degree 0, a -degree 1 and x -degree $N + 1$. Set $s = sl(B)$. Taking the homology with respect to d_{mf} , the above short exact sequence gives the following

long exact sequence.

$$\begin{array}{ccccccc}
& & & & & & \dots \\
& & & \swarrow & & & \\
H^{s-1,*,*,k-N-1}(\mathcal{C}_N(B), d_{mf})\{1\}_a & \xrightarrow{a} & H^{s-1,*,*,k-N-1}(\mathcal{C}_N(B), d_{mf})\{-1\}_a & \xrightarrow{\pi_0} & H^{s-1,*,*,k-N-1}(\mathcal{C}_N(B), d_{mf})\{-1\}_a & & \\
& & \swarrow & & \swarrow & & \\
H^{s,*,*,k}(\mathcal{C}_N(B), d_{mf}) & \xrightarrow{a} & H^{s,*,*,k}(\mathcal{C}_N(B), d_{mf})\{-2\}_a & \xrightarrow{\pi_0} & H^{s,*,*,k}(\mathcal{C}_N(B), d_{mf})\{-2\}_a & & \\
& & \swarrow & & \swarrow & & \\
H^{s-1,*,*,k+N+1}(\mathcal{C}_N(B), d_{mf})\{-1\}_a & \xrightarrow{a} & H^{s-1,*,*,k+N+1}(\mathcal{C}_N(B), d_{mf})\{-3\}_a & \longrightarrow & \dots & &
\end{array}$$

Following the notations in Subsection 3.3, we denote by $F^{\varepsilon,i,*,k}$ the free part of $H^{\varepsilon,i,*,k}(\mathcal{C}_N(B), d_{mf})$ and by $T^{\varepsilon,i,*,k}$ the torsion part of $H^{\varepsilon,i,*,k}(\mathcal{C}_N(B), d_{mf})$. By Corollary 3.10 and the normalization of the local chain complexes (2.7) and (2.8), we know that $F^{s-1,i,*,k} \cong 0$ and $T^{\varepsilon,i,*,k}$ is a direct sum of components of the form $\mathbb{Q}[a]/(a)\{*\}_a$. So the above long exact sequence breaks into two exact sequences:

$$(4.1) \quad 0 \rightarrow H^{s-1,*,*,k-N-1}(\mathcal{C}_N(B), d_{mf})\{-1\}_a \xrightarrow{\pi_0} H^{s-1,*,*,k-N-1}(\mathcal{C}_N(B), d_{mf})\{-1\}_a \rightarrow T^{\varepsilon,*,*,k} \rightarrow 0$$

and

$$(4.2) \quad 0 \rightarrow F^{s,*,*,k} \rightarrow H^{s,*,*,k}(\mathcal{C}_N(B), d_{mf})\{-2\}_a \xrightarrow{\pi_0} H^{s,*,*,k}(\mathcal{C}_N(B), d_{mf})\{-2\}_a \rightarrow H^{s-1,*,*,k+N+1}(\mathcal{C}_N(B), d_{mf})\{-1\}_a \rightarrow 0.$$

Applying Lemma 4.2 to the exact sequence (4.1), we get that

$$H^{s-1,i,*,k}(\text{cone}(H(\mathcal{C}_N(B), d_{mf}) \xrightarrow{\pi_0} H(\mathcal{C}_N(B), d_{mf})), d_\chi)\{-1\}_a \cong H^{i-1}(T^{s,*,*,k+N+1}, d_\chi).$$

By [9, Theorem 1.5],

$$\mathcal{H}_N^{s-1,i,*,k}(B_-) \cong H^{s-1,i,*,k}(\text{cone}(H(\mathcal{C}_N(B), d_{mf}) \xrightarrow{\pi_0} H(\mathcal{C}_N(B), d_{mf})), d_\chi)\{-2\}_a.$$

So

$$\mathcal{H}_N^{s-1,i,*,k}(B_-) \cong H^{i-1}(T^{s,*,*,k+N+1}, d_\chi)\{-1\}_a.$$

By Lemmas 3.11 and 3.12, there is a long exact sequence

$$\dots \rightarrow H^i(T^{\varepsilon,*,*,k}) \rightarrow \mathcal{H}_N^{\varepsilon,i,*,k}(B) \rightarrow H_N^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s\}_a \rightarrow H^{i+1}(T^{\varepsilon,*,*,k}) \rightarrow \dots$$

Thus, we have a long exact sequence

$$\dots \rightarrow \mathcal{H}_N^{s-1,i,*,k}(B_-) \rightarrow \mathcal{H}_N^{s,i-1,*,k+N+1}(B)\{-1\}_a \rightarrow H_N^{s,i-1,k+N+1}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s-1\}_a \rightarrow \mathcal{H}_N^{s-1,i+1,*,k}(B_-) \rightarrow \dots$$

This establishes the long exact sequence (1.1).

Now apply Lemma 4.3 to the exact sequence (4.2). Using also the fact that

$$\mathcal{H}_N^{s,i,*,k}(B_-) \cong H^{s,i,*,k}(\text{cone}(H(\mathcal{C}_N(B), d_{mf}) \xrightarrow{\pi_0} H(\mathcal{C}_N(B), d_{mf})), d_\chi)\{-2\}_a,$$

we get a long exact sequence

$$\dots \rightarrow H^i(F^{s,*,*,k}) \rightarrow \mathcal{H}_N^{s,i,*,k}(B_-) \rightarrow \mathcal{H}_N^{s,i-1,*,k+N+1}(B)\{-1\}_a \rightarrow H^{i+1}(F^{s,*,*,k}) \rightarrow \dots$$

By Lemma 3.12, $H^i(F^{s,*,*,k}) \cong H_N^{s,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s\}_a$, which is a free $\mathbb{Q}[a]$ -module. From [4], we know that $H_N^{s-1,*,*,k}(B) \cong 0$. So, by Theorem 1.4, $\mathcal{H}_N^{s-1,i-1,*,k+N+1}(B)$ is a torsion $\mathbb{Q}[a]$ -module. Thus, the above long exact sequence breaks into the following short exact sequence.

$$0 \rightarrow H_N^{s,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s\}_a \rightarrow \mathcal{H}_N^{s,i,*,k}(B_-) \rightarrow \mathcal{H}_N^{s-1,i-1,*,k+N+1}(B)\{-1\}_a \rightarrow 0.$$

This establishes the short exact sequence (1.2). □

4.3. Transverse unknots. We are now ready to prove Corollary 1.7. We start by a simple algebraic observation.

Lemma 4.4. *Let $\mathcal{F} = \bigoplus_{l=0}^{N-1} \mathbb{Q}[a] \langle 1 \rangle \{-1, -N+1+2l\}$ be as defined in Lemma 1.7.*

- (1) *Assume $f : \mathcal{F} \rightarrow \mathcal{F}$ is an injective homogeneous homomorphism of a -degree 2 and preserving other gradings. Then $\text{coker } f \cong \mathcal{F}/a\mathcal{F}$.*
- (2) *Assume $g : \mathcal{F} \rightarrow \mathcal{F}$ is an injective homogeneous homomorphism preserving all gradings. Then g is an isomorphism.*

Proof. The proofs for the two parts are very similar. We only include here the proof for Part (1) and leave Part (2) for the reader.

Denote by 1_l the “1” in $\mathbb{Q}[a] \langle 1 \rangle \{-1, -N+1+2l\}$. Then, since f is an injective homogeneous homomorphism of a -degree 2 and preserves the x -grading, we know that $f(1_l) = \lambda_l a 1_l$ for some $\lambda_l \in \mathbb{Q} \setminus \{0\}$. The lemma follows from this. \square

Proof of Corollary 1.7. Setting $b = 1$ in Lemma 3.6, we get that $\mathcal{H}_N(U_0) \cong \mathcal{F} \oplus \mathcal{T}$.

For $m = 1$, the exact sequences in Theorem 1.6 are non-vanishing at only two locations:

$$(4.3) \quad 0 \rightarrow \mathcal{H}_N^{0,1,*}(U_1) \rightarrow \mathcal{H}_N^{1,0,*}(U_0)\{-1, -N-1\} \rightarrow H_N^{1,0,*}(U_0) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{-2, -N-1\} \rightarrow \mathcal{H}_N^{0,2,*}(U_1) \rightarrow 0,$$

$$(4.4) \quad 0 \rightarrow H_N^{1,0,*}(U_0) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{-1\}_a \rightarrow \mathcal{H}_N^{1,0,*}(U_1) \rightarrow 0.$$

Recall that, from [4], we know that $H_N(U_m) \cong H_N(U_0) \cong \bigoplus_{l=0}^{N-1} \mathbb{Q} \langle 1 \rangle \{-N+1+2l\}_x$. So

$$(4.5) \quad H_N(U_m) \otimes_{\mathbb{Q}} \mathbb{Q}[a] \cong H_N(U_0) \otimes_{\mathbb{Q}} \mathbb{Q}[a] \cong \bigoplus_{l=0}^{N-1} \mathbb{Q}[a] \langle 1 \rangle \{0, -N+1+2l\} \cong \mathcal{F}\{1\}_a.$$

Also, by Remark 1.5, $\mathcal{H}_N^{0,1,*}(U_1)$ is a torsion $\mathbb{Q}[a]$ -module. So exact sequence (4.3) breaks into

$$(4.6) \quad 0 \rightarrow \mathcal{H}_N^{0,1,*}(U_1) \rightarrow \mathcal{T}\{-1, -N-1\} \rightarrow 0,$$

$$(4.7) \quad 0 \rightarrow \mathcal{F}\{-1, -N-1\} \rightarrow \mathcal{F}\{-1, -N-1\} \rightarrow \mathcal{H}_N^{0,2,*}(U_1) \rightarrow 0.$$

Thus, we have $\mathcal{H}_N^{0,1,*}(U_1) \cong \mathcal{T}\{-1, -N-1\}$ and, by Part (2) of Lemma 4.4, $\mathcal{H}_N^{0,2,*}(U_1) \cong 0$. Also, using exact sequence (4.4), we have $\mathcal{H}_N^{1,0,*}(U_1) \cong \mathcal{F}$. Putting everything together, we have $\mathcal{H}_N(U_1) \cong \mathcal{F} \oplus \mathcal{T} \langle 1 \rangle \{-1, -N-1\} \| 1 \|$

Next, assume the corollary is true for U_m for some $m \geq 1$. We prove that the corollary is true for U_{m+1} .

By (4.5), $H_N^{\varepsilon,i,*}(U_m) \otimes_{\mathbb{Q}} \mathbb{Q}[a] \cong \begin{cases} \mathcal{F}\{1\}_a & \text{if } \varepsilon = 1 \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$ So the exact sequences in Theorem 1.6 break into

$$(4.8) \quad 0 \rightarrow \mathcal{F}\{-2m\}_a \rightarrow \mathcal{H}_N^{1,0,*}(U_{m+1}) \rightarrow 0$$

$$(4.9) \quad 0 \rightarrow \mathcal{H}_N^{0,1,*}(U_{m+1}) \rightarrow \mathcal{F}\{-2m+1, -N-1\} \rightarrow \mathcal{F}\{-2m-1, -N-1\} \rightarrow \mathcal{H}_N^{0,2,*}(U_{m+1}) \rightarrow 0,$$

$$(4.10) \quad 0 \rightarrow \mathcal{H}_N^{l+1,l+2,*}(U_{m+1}) \rightarrow \mathcal{F}/a\mathcal{F}\{-2m+l-1, -(l+1)(N+1)\} \rightarrow 0, \text{ for } l = 1, \dots, m-1,$$

$$(4.11) \quad 0 \rightarrow \mathcal{H}_N^{m-1,m+1,*}(U_{m+1}) \rightarrow \mathcal{T}\{-m-1, -(m+1)(N+1)\} \rightarrow 0.$$

Exactness of (4.8) gives us

$$\mathcal{H}_N^{1,0,*}(U_{m+1}) \cong \mathcal{F}\{-2m\}_a.$$

Exactness of (4.10) and (4.11) give us

$$\begin{aligned} \mathcal{H}_N^{l+1,l+2,*}(U_{m+1}) &\cong \mathcal{F}/a\mathcal{F}\{-2m+l-1, -(l+1)(N+1)\}, \\ \mathcal{H}_N^{m-1,m+1,*}(U_{m+1}) &\cong \mathcal{T}\{-m-1, -(m+1)(N+1)\}. \end{aligned}$$

Finally, we look at exact sequence (4.9). By Remark 1.5, $\mathcal{H}_N^{0,1,*}(U_{m+1})$ is a torsion $\mathbb{Q}[a]$ -module. This implies that $\mathcal{H}_N^{0,1,*}(U_{m+1}) \cong 0$ and we have a short exact sequence

$$0 \rightarrow \mathcal{F}\{-2m+1, -N-1\} \rightarrow \mathcal{F}\{-2m-1, -N-1\} \rightarrow \mathcal{H}_N^{0,2,*}(U_{m+1}) \rightarrow 0.$$

Applying Part (1) of Lemma 4.4 to the above short exact sequence, we get

$$\mathcal{H}_N^{0,2,\star,\star}(U_{m+1}) \cong \mathcal{F}/a\mathcal{F}\{-2m-1, -N-1\}.$$

Now putting everything together, we have that

$$\begin{aligned} \mathcal{H}_N(U_{m+1}) &\cong \mathcal{F}\{-2((m+1)-1), 0\} \oplus \mathcal{T}\langle m+1 \rangle \{-(m+1), -(m+1)(N+1)\} \|m+1\| \\ &\oplus \bigoplus_{l=1}^{(m+1)-1} \mathcal{F}/a\mathcal{F}\langle l \rangle \{-2(m+1)+l, -l(N+1)\} \|l+1\|. \end{aligned}$$

This shows that the corollary is true for U_{m+1} too. □

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